THEORY OF STATIONARY WAVES IN AN INHOMOGENEOUS FLUID

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PMM Vol. 29, No. 3, 1965, pp. 440-452

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(Received February 18, 1965)

Two possible formulations of the problem of the motion of an inhomogeneous fluid with a free boundary are examined. In formulation A, distributions of density and of the horizontal component of the velocity vector are given as piecewise smooth functions of the ordinate for a certain cross section of the flow. In formulation B the distribution of the density and of the average vorticity are given along streamlines. Problems A and B are reduced to certain functional equations and these equations are investigated for values of the parameter which are close to the critical values.

1. Formulation of problems A and B. The two-dimensional stationary flow of an ideal incompressible inhomogeneous fluid above a rectilinear bed is examined. The upper boundary of the fluid is free and n interfaces exist on which the density and tangential component of the velocity vector suffer discontinuities of the first kind. The y-axis is taken vertically upwards, and the x-axis along the bed of the channel. The lines of separation are not known and must be determined in solving the problem. Let $\mathbf{v}$ be the velocity vector, p the hydrodynamic pressure, ρ the density, g the acceleration due to gravity, and $\mathbf{a} = \mathbf{v}\sqrt{\rho}$. Then in each of the regions

$$T_k \;(-\infty < x < +\infty,\; Y_{k-1}(x) < y < Y_k(x))$$

the equations of motion with nondimensional variables can be written in the form [1]

$$\operatorname{div} \mathbf{a} = 0, \quad \mathbf{a}\cdot\nabla\rho = 0, \quad (\mathbf{a}\nabla)\,\mathbf{a} = -\nu\rho\mathbf{y}^\circ - \nabla p \qquad (\nu = gH/c^2) \quad (1.1)$$

The characteristic velocity and characteristic depth for different problems are selected in different ways. If we assume that in the transition through interfaces the pressure must change continuously, then the boundary conditions can be written in the form

$$a_y(x, 0) = 0; \qquad \mathbf{a}\cdot\mathbf{n} = 0, \quad [p]_k = 0 \quad \text{for} \quad y = Y_k(x) \quad (k = 1, \ldots, n) \quad (1.2)$$

where $\mathbf{n}$ is the external normal and

$$[p]_n = p\ (x,\ Y_n - 0), \quad [p]_k = p\ (x,\ Y_k - 0) - p\ (x, Y_k + 0) \quad (k = 1, \ldots, n-1)$$

Since the fluid is swirling and inhomogeneous it is necessary to assign the distribution of density and vorticity in some way. Prior to presenting possible formulations here, some general properties of the system of equations (1.1) and (1.2) are investigated. The first of the equations (1.1) enables a stream function to be introduced for the vector $\mathbf{a}$

$$a_x = \partial\psi / \partial y, \qquad a_y = -\partial\psi / \partial x$$

By virtue of the boundary conditions (1.2) the bed and the free boundary must represent streamlines. Since in each of the regions T_k, the function $\psi\ (x,\ y)$ is known with an accuracy to within an arbitrary constant, these arbitrary constants can be selected such that the function $\psi\ (x,\ y)$ will be continuous in the region occupied by the fluid. The properties of the family of streamlines are expressed by the following theorem.

Theorem 1.1. If the functions $Y_k(x)$ are continuously differentiable, if vector function $\mathbf{a}\ (x,\ y)$ is continuously differentiable in regions T_k and suffers discontinuities of the first kind in a transition across interfaces, and if the following inequalities are satisfied

$$0 < A_1 \leqslant a_x\ (x,\ y) \leqslant A_2, \qquad \max\ \{|\ a_y\ |,\ |\ \nabla a_x\ |,\ |\ \nabla a_y\ |\} < A_3$$

then the streamlines cannot be closed, they cannot originate or terminate on lines of separation, and the equation for the family of streamlines can be solved with respect to y.

Proof of theorem 1.1 is omitted since it is a trivial consequence of the theorem of existence and the uniqueness of the solution of the Cauchy problem for an ordinary differential equation of a streamline $dx\ /\ a_x\ (x,\ y) = dy\ /\ a_y\ (x,\ y)$.

It is not difficult to show that elements located on one streamline have the following constant density and total energy [2]

$$\rho\ (x,\ y) = R\ (\psi), \qquad {}^1/_2\, a^2 + p + \nu R\ (\psi)\ y = h\ (\psi) \tag{1.3}$$

and the function $\psi\ (x,\ y)$ must be a solution of the equation

$$\triangle\psi + \nu R'\ (\psi)\ y = h'\ (\psi) \tag{1.4}$$

Here $R\ (\psi)$ and $h\ (\psi)$ are arbitrary functions. Let us assume that the units of measurement are selected in such a way that the flux corresponding to the vector $\mathbf{a}$ through the cross section of the channel is equal to one. If the pressure is eliminated from the boundary conditions (1.2) by means of equations (1.3), then these boundary conditions can be written in the form (P_k are unknown quantities):

$$\begin{gathered} \psi\ (x,\ Y_k\ (x)) = P_k, \quad Y_0 \equiv 0, \quad 0 = P_0 < P_1 < P_2 < \ldots < P_n = 1 \\ [{}^1/_2\, (\nabla\psi)^2 + \nu R\ (\psi)\ y - h\ (\psi)]_k = 0 \qquad (k = 1, \ldots, n) \end{gathered} \tag{1.5}$$

Equations (1.4) and (1.5) contain two arbitrary functions $h(\psi)$ and $R(\psi)$ and $n-1$ arbitrary quantities $P_1, \ldots, P_{n-1}$. For definition of the problem it is necessary to give additional physical conditions which characterize the density and vorticity distribution.

In formulation A the ordinates of the interfaces and also the distribution of the density and of the horizontal component of the velocity vector are given on the channel cross section $x = 0$, which is the axis of symmetry. Then the ordinate H of the free boundary, the average density ρ^0, and the flux P of the vector **a** through a given cross section can be determined. We take the quantity H as a characteristic dimension, and the quantity $c = P / H \sqrt{\rho^0}$, as a characteristic velocity which can also be called the velocity of wave propagation. Then the flux of the vector **a** will be equal to unity in nondimensional variables. The ordinates of the interfaces and the distribution of the density and of the horizontal component of the velocity vector are given in the form

$$Y_k(0) = h_k, \qquad h_0 \equiv 0 < h_1 < \ldots < h_n \equiv 1 \qquad (k = 1, \ldots, n)$$
$$\rho(0, y) = \rho_0(y), \qquad a_x(0, y) = q(y) \tag{1.6}$$

Here $h_1, \ldots, h_{n-1}$ are known quantities while $\rho_0(y)$ and $q(y)$ are known piecewise functions having discontinuities of the first kind at points $h_1, \ldots, h_{n-1}$ satisfying the following conditions

$$\rho_0(y) \geqslant R_0 > 0, \qquad d\rho_0 / dy \leqslant 0, \qquad q(y) \geqslant Q > 0 \tag{1.7}$$

Bounds on the functions $\rho_0(y)$ are quite natural from a physical point of view. They indicate that the heavy layers of liquid are located lower than the lighter ones. The bounds on the function $q(y)$ are more restrictive and must provide that the conditions of theorem 1.1 are satisfied. Let E_A denote the set of pairs of the functions $[\rho_0(y), q(y)]$, which have discontinuities of the first kind at the points $h_1, \ldots, h_{n-1}$, which are twice continuously differentiable at $h_{k-1} < \eta < h_k$ and which satisfy conditions (1.7).

It will be shown that the independent function $R(\psi)$ can be determined. For this, conditions (1.6) are rewritten in the form

$$R[\psi(0, y)] = \rho_0(y), \qquad (\partial\psi / \partial y)_{x=0} = q(y) \tag{1.8}$$

Integrating the second condition (1.8) we obtain that the function

$$\varphi(y) = \psi(0, y) = \int_0^y q(t)\, dt$$

will be monotonic and continuous and it therefore has a continuous and monotonic inverse function $y = \eta(\phi)$, where $\eta(0) = 0$, $\eta(P_k) = h_k$. Substituting the quantity $y = \eta(\phi)$ into the right-hand side of the first of equations (1.8), we obtain that $R(\varphi) = \rho_0[\eta(\varphi)]$, and consequently the function $R(\psi)$ is determined.

Assuming that the conditions of theorem 1.1 are satisfied, a change of variables is made in equation (1.4) and the boundary conditions (1.5); x and η are taken as independent variables, and y is taken as the unknown function. Omitting simple transformations we obtain

$$\frac{1}{2}\frac{\partial}{\partial\eta}\left[q^2(\eta)\,\frac{1+y_x^{\,2}}{y_\eta^{\,2}}\right]-q^2(\eta)\,\frac{\partial}{\partial x}\left(\frac{y_x}{y_\eta}\right)+\nu\rho_0'(\eta)\,y=\hbar(\eta),\quad [y]_k=0 \quad (k=1,\ldots,n-1)$$

$$y(x,0)=0,\quad y(0,\eta)=\eta,\quad \left[\frac{1}{2}q^2\,\frac{1+y_x^{\,2}}{y_\eta^{\,2}}+\nu\rho_0(\eta)\,y-\hbar\right]_k=0\ (k=1,\ldots,n) \tag{1.9}$$

In order to eliminate the function $\hbar(\eta)$, from equations (1.9) we write

$$y=\eta+\int_0^x w(x,\eta)\,dx\equiv\eta+\mathrm{S}w \tag{1.10}$$

Then the boundary condition $y(0,\eta)=\eta$ will be satisfied automatically and to determine the function $w(x,\eta)$ it will be necessary to solve the boundary value problem

$$\mathrm{M}w\equiv\frac{\partial}{\partial\eta}\left[q^2(\eta)\,\frac{\partial w}{\partial\eta}\right]+q^2(\eta)\,\frac{\partial^2 w}{\partial x^2}=\nu\rho_0'(\eta)\,w+\operatorname{div}(q^2\Phi w),\qquad [w]_k=0 \quad (k=1,\ldots,n-1)$$

$$w(x,0)=0,\quad \left[q^2(\eta)\,\frac{\partial w}{\partial\eta}-\nu\rho_0(\eta)\,w-q^2\Phi_2 w\right]_k=0 \qquad (k=1,\ldots,n) \tag{1.11}$$

where Φw is the following nonlinear operator

$$\Phi w=(\Phi_1 w,\,\Phi_2 w),\qquad \Phi_1 w=\frac{ww_\eta}{(1+\mathrm{S}w_\eta)^2}+\frac{w_x\mathrm{S}w_\eta}{1+\mathrm{S}w_\eta}$$

$$\Phi_2 w=\frac{ww_x}{(1+\mathrm{S}w_\eta)^2}+\frac{3\mathrm{S}w_\eta+3(\mathrm{S}w_\eta)^2+(\mathrm{S}w_\eta)^3-w^2}{(1+\mathrm{S}w_\eta)^3}\,w_\eta \tag{1.12}$$

In this fashion problem A is reduced to the solution of a nonlinear boundary value problem (1.11). We will look for periodic (with fixed period $2L$) odd solutions of this problem. There is always the trivial solution $w(x,\eta)=0$ which corresponds to one-dimensional flow. The question arises about the existence of two-dimensional flows which are close to one-dimensional flow. Mathematically this is the problem of the branching of solutions of a nonlinear equation.

We also note that for one-dimensional flow, the conditions of theorem 1.1 are satisfied. Consequently they will also be satisfied for two-dimensional flow which is close to one-dimensional. From the solution of problem A, results for potential flow do not follow because for potential flow it is not permissible to prescribe the distribution of velocity arbitrarily.

In formulation B we will consider the average depth of layers and the distribution of the density and of the average vorticity of vector **a** along streamlines as given

$$H_k = \frac{1}{L}\int_0^L Y_k(x)\,dx, \qquad Y_0 \equiv 0 \qquad (k = 1, \ldots, n)$$
$$\rho(x, y) = R(\psi), \qquad \sigma(\psi) = -\frac{1}{L}\int_0^L \Delta\psi\, dx \tag{1.13}$$

We also consider the discharge of liquid through the cross section of layers $P_1, \ldots, P_n$. as given. The units of measurement are selected in such a way that the average depth of the channel, the flow of liquid through the cross section, and the average density ρ are equal to unity. A substitution of variables is made in equation (1.4) and boundary conditions (1.5); x and ψ are taken as independent variables and y is taken as the unknown function. If advantage is taken of conditions (1.13), then we obtain that the function $y(x, \psi)$ must be a solution of the boundary value problem

$$\frac{\partial}{\partial\psi}\left(\frac{1 + y_x^2}{y_\psi^2}\right) - \frac{\partial}{\partial x}\left(\frac{y_x}{y_\psi}\right) + \nu R'(\psi)\left[y - \frac{1}{2L}\int_{-L}^{L} y(x, \psi)\,dx\right] = -\sigma(\psi)$$

$$y(x, 0) = 0, \qquad \frac{1}{L}\int_0^L y(x, P_k)\,dx = H_k, \quad [y]_k = 0 \quad (k = 1, \ldots, n-1) \tag{1.14}$$

$$\left[\frac{1}{2}\left(\frac{1 + y_x^2}{y_\psi^2} - \frac{1}{2L}\int_{-L}^{L} \frac{1 + y_x^2}{y_\psi^2}\,dx\right) + \nu R(\psi)(y - H_k)\right]_k = 0 \qquad (k = 1, \ldots, n)$$

We require that the function $R(\psi)$ satisfy the conditions

$$R(\psi) \geqslant R_0 > 0, \qquad R'(\psi) \leqslant 0, \qquad [R]_k \geqslant 0 \tag{1.15}$$

As before these conditions indicate that the density increases with depth. The bounds of the function $\sigma(\psi)$ are derived from the condition that the boundary value problem (1.14) must allow solutions which are independent of x (one-dimensional flow). The boundary value problem corresponding to one-dimensional flow has the form

$$y = \eta(\psi), \quad \frac{d}{d\psi}\frac{1}{\eta_\psi^2} = -\sigma(\psi), \quad [\eta]_k = 0, \quad \eta(P_k) = H_k, \quad \eta(0) = 0 \quad (k = 1, \ldots, n)$$

Integrating the second equation we obtain

$$\eta = H_{k-1} + \int_{P_{k-1}}^{\psi} \frac{d\psi}{\sqrt{c_k - u(\psi)}}, \qquad u(\psi) = 2\int_{P_{k-1}}^{\psi} \sigma(v)\,dv \qquad (k = 1, \ldots, n)$$

The arbitrary constant c_k can be determined from the condition $\eta(P_k) = H_k$ in the case when

$$H_k - H_{k-1} \leqslant \int_{P_{k-1}}^{P_k} \frac{d\psi}{\sqrt{u_0^k - u(\psi)}}, \quad u_0^k = \max u(\psi) \quad (P_{k-1} \leqslant \psi \leqslant P_k, \ k = 1, \ldots, n)$$

It is convenient to make a change of variables in equations (1.14), taking $\eta(\psi)$ as the new independent variable and

$$w(x, \eta) = y(x, \psi) - \eta(\psi)$$

as the unknown function. Introducing further the notation

$$f_c(\eta) = \frac{1}{2L} \int_{-L}^{L} f(x, \eta)\, dx, \quad f_g(x, \eta) = f(x, \eta) - f_c(x, \eta), \quad q(\eta) = \frac{d\psi}{d\eta} > 0$$

it is possible to reduce the boundary value problem (1.14) to a solution of the following connected boundary value problems for w_c and w_g:

$$\frac{d}{d\eta}\left[q^2(\eta)\frac{dw_c}{d\eta}\right] = \frac{d}{d\eta}[q^2(\eta)(\Phi_2 w)_c], \quad w_c(0) = w_c(H_1) = \ldots = w_c(H_n) = 0 \qquad (1.16)$$

$$\begin{gathered} \mathrm{M}w_g = \nu\rho'(\eta) w_g + \operatorname{div}[q^2(\eta)(\Phi w)_g], \quad w = w_c + w_g, \quad [w_g]_k = 0 \\ (k = 1, \ldots, n-1) \\ \left[q^2(\eta)\frac{\partial w_g}{\partial \eta} - \nu\rho w_g - q^2(\eta)(\Phi_2 w)_g\right]_k = 0 \quad (k = 1, \ldots, n) \end{gathered} \qquad (1.17)$$

where the operator M is determined from formula (1.11) and

$$\Phi_2 w = \frac{1}{2}\frac{3w_\eta^2 + 2w_\eta^3 + w_x^2}{(1 + w_\eta)^2}, \qquad \Phi_1 w = \frac{w_x w_\eta}{1 + w_\eta}, \qquad \Phi = (\Phi_1, \Phi_2)$$

In this fashion, problem B is reduced to the boundary value problem (1.16) and (1.17). We will try to find even periodic solutions of this problem with period $2L$ which are different from the trivial solution.

Problem B has a fairly general character. If for example $\sigma(\psi) \equiv 0$ and $\rho(\psi)$ is a piecewise constant function, then problem B describes the potential flows of a multilayer fluid. A particular case of the latter problem ($n = 2$) was examined by N.E. Kochin. Since methods for the examination of problems A and B are not substantially different, the analysis will be carried out for problem A, and the corresponding results for problem B will be formulated without detailed investigation.

2. Linear theory. Neglecting nonlinear terms in (1.11) we arrive at the following mathematical eigenvalue problem:

$$\begin{gathered} \mathrm{M}w = \nu\rho' w, \quad w(x, 0) = w(0, \eta) = w(L, \eta) = 0 \\ [w]_k = 0, \quad [q^2 w_\eta - \nu\rho w]_k = 0 \end{gathered}$$

If this problem is solved by separating the variables, then the eigenvalues and eigenfunctions will have the form

$$\nu_{mk} = \nu_m\left(\frac{k\pi}{L}\right), \qquad w_{mk}(x, \eta) = \frac{1}{\sqrt{L}} u_m\left(\eta, \frac{k\pi}{L}\right) \sin\frac{k\pi x}{L} \tag{2.1}$$

Here $\nu_m(\lambda)$ and $u_m(\eta, \lambda)$ are eigenvalues and eigenfunctions of the following boundary value problem for the ordinary equation:

$$\frac{d}{d\eta}\left(q^2 \frac{du}{d\eta}\right) - \lambda^2 q^2 u = \nu\rho' u, \quad u(0, \lambda) = 0, \quad [u]_k = 0, \quad \left[q^2 \frac{du}{d\eta} - \nu\rho u\right]_k = 0 \tag{2.2}$$

which in turn is reduced to the problem of the minimum of the functional

$$\chi_\lambda(u) = \left\{\sum_{k=1}^{n} [\rho]_k \,|\, u(h_k)\,|^2 - \int_0^1 \rho'(\eta)\,|\,u(\eta)\,|^2 d\eta\right\}^{-1} \int_0^1 q^2(\eta)\left[\left|\frac{du}{d\eta}\right|^2 + \lambda^2\,|\,u\,|^2\right] d\eta$$

The variational problem can be analyzed by straightforward methods [3]. As a result we arrive at the following theorem.

Theorem 2.1. If the functions $\rho(\eta)$ and $q(\eta)$ satisfy conditions (1.7), then all eigenvalues of the boundary value problem (2.2) are simple and real. If the measure of the set, on which $\rho'(\eta) \not\equiv 0$, is different from zero, then the eigenvalues form a denumerable set which does not have points of increased density at a finite distance. If however $\rho'(\eta) = 0$ almost everywhere, then there will be a finite number of eigenvalues.

It is clear from expressions (2.1) that in investigating the properties of eigenvalues and eigenfunctions it is sufficient to limit oneself to the case $k = 1$, since all other eigenfunctions and eigenvalues are obtained by dividing the half-period into an integral number of parts. The question of spectrum multiplicity is very important. Few cases are known where this problem can be solved completely. In the case under examination it is only possible to assert that the first eigenvalue is simply due to the fact that min $\chi_\lambda(u)$ is an increasing function of the parameter λ.

In order to reduce the problem of the existence of periodic solutions of the boundary value problem (1.11) to functional equations, it is necessary to know the properties of the solution of the inhomogeneous linear equation

$$\begin{gathered} Mw - \nu\rho' w = \operatorname{div}(q^2 \mathbf{f}), \qquad \mathbf{f} = (f_1, f_2), \quad [w]_k = 0 \quad (k = 1, \ldots, n-1) \\ w(x, 0) = w(0, \eta) = w(L, \eta) = 0, \qquad [q^2 w_\eta - \nu\rho w - q^2 f_2]_k = 0 \quad (k = 1, \ldots, n) \end{gathered} \tag{2.3}$$

Here f_1 and f_2 are some piecewise smooth functions which have discontinuities of the first kind at points $h_1, h_2, \ldots, h_{n-1}$, while $f_1(x, \eta)$ is an even periodic function and $f_2(x, \eta)$ is odd,

$$f_2(0, \eta) = f_2(L, \eta) = f_{1x}(0, \eta) = f_{1x}(L, \eta) = 0$$

In the following it will be convenient to make use of the terminology of functional analysis. Let D_k be a rectangle $(0 \leqslant x \leqslant L, \ h_{k-1} \leqslant \eta \leqslant h_k)$, and D be a rectangle

$(0 \leqslant x \leqslant L,\ 0 \leqslant \eta \leqslant 1)$ and let H_m^k be the Hoelder space of functions which in the rectangle D_k have derivatives of order m satisfying Hoelder's condition with the index α $(0 < \alpha < 1)$. Let C be the space of functions continuous in the rectangle D. We denote by H_m the Banach space of continuous functions defined in the rectangle D and such that $w \in H_m^k$ for $(x, \eta) \in D_k$. The norm is defined in the following way:

$$\| w \|_{H_m} = \| w \|_c + \| w \|_{H_m^1} + \ldots + \| w \|_{H_m^n}$$

Let B_m be the space of the pair of functions $\mathbf{f} = (f_1, f_2)$, where $f_1 \in H_m$, and $f_2 \in H_m$ and $f_1(x, \eta)$ is an even periodic function (with period $2L$) and $f_2(x, \eta)$ is uneven.

Theorem 2.2. If $\nu = \nu_i$ is an l_i-fold eigenvalue of the homogeneous problem, and $z_{i1}(x, \eta), \ldots z_{il_i}(x, \eta)$ are eigenfunctions corresponding to this eigenvalue, and $(\rho, q) \in E_A$, $\mathbf{f} \in B_1$, then the inhomogeneous boundary value problem (2.3) can be solved if and only if the following conditions are satisfied

$$\iint\limits_D q^2(\eta)\, \mathbf{f} \cdot \nabla z_{ik}\, dx\, d\eta = 0 \qquad (k = 1, \ldots, l_i) \tag{2.4}$$

The solution can then be written in the form

$$w = \mathrm{A}\mathbf{f} + \sum_{k=1}^{l_i} c_k z_k(x, \eta)$$

Here A is a linear operator acting from space B_1 into space H_2, c_k are arbitrary numbers, and

$$\| \mathrm{A}\mathbf{f} \|_{H_2} \leqslant \text{const}\, \{\| f_1 \|_{B_1} + \| f_2 \|_{B_1}\}$$

Proof of theorem 2.2. is not very simple, so we will break it up into a series of lemmas.

Let us examine the Hilbert space of functions which can be integrated with a square in D and such that the function $w(x, h_k)$ can be integrated with a square with respect to x for $0 \leqslant x \leqslant L$, if $[\rho]_k \neq 0$. We note that it is automatically known that $[\rho]_n \neq 0$. The scalar product in H_ρ is determined in the following way

$$(w_1, w_2)_{H_\rho} = \sum_{k=1}^{n}{}' [\rho]_k \int_0^L w_1(x, h_k)\, \overline{w_2(x, h_k)}\, dx - \int_0^1 \int_0^L \rho'(\eta)\, w_1(x, \eta)\, \overline{w_2(x, \eta)}\, dx\, d\eta$$

All functions which differ only on the set $E_\rho(\rho' = 0)$ will be considered identical. Let us examine the subset H_0 of functions from B_2 which become zero in the boundary layer D_ε $(0 \leqslant x \leqslant \varepsilon,\ 0 \leqslant \eta \leqslant 1;\ \ 0 \leqslant \eta \leqslant \varepsilon,\ 0 \leqslant x \leqslant L;\ \ L - \varepsilon \leqslant x \leqslant L,\ 0 \leqslant \eta \leqslant 1)$. On the set H_0 we introduce the following scalar product

$$(w_1, w_2)_{H_0} = \int_0^1 \int_0^L q^2(\eta) \left[\frac{\partial w_1}{\partial x} \overline{\frac{\partial w_2}{\partial x}} + \frac{\partial w_1}{\partial \eta} \overline{\frac{\partial w_2}{\partial \eta}} \right] dx\, d\eta$$

The Hilbert space H_0 will be incomplete. If it is completed then some subspace H of Sobolev's space $W_2^{(1)}$ is obtained for functions which have the first generalized derivatives summable with a square [3]. Problem (2.3) is easily reduced to a problem on the minimum of the functional

$$F(w) = \|w\|_{H_0}^2 - \nu_i \|w\|_{H_\rho}^2 - 2\,\mathrm{Re}\left[\iint_D q^2\,\mathbf{f}\cdot\nabla w\,dx\,d\eta\right] \tag{2.5}$$

The homogeneous linear problem corresponds to the problem on the minimum of the functional $\chi(w) = \|w\|_H^2 / \|w\|_{H_\rho}^2$. Eigenvalues and eigenfunctions of the homogeneous problem were already defined by equations (2.1). If the eigenvalues are arranged in the order of increasing $(\nu_1 < \nu_2 < \ldots < \nu_n < \ldots)$, then, as is well known from the classical theory of the minimum of the quadratic functional [3],

$$\nu_1 = \min \chi(w), \quad w \in H$$

$$\nu_{k+1} = \min \chi(w), \quad w \in H, \quad (w, z_{ij})_{H_\rho} = 0 \quad (i = 1, \ldots, k;\ j = 1, \ldots, l_i)$$

It is also noted that functions orthogonal to corresponding functions in H_ρ, will also be orthogonal to them in H. This follows from the equation in variations for the functional $X(w)$

$$(z_{ij}, \psi)_H - \nu_i (z_{ij}, \psi)_{H_\rho} = 0, \quad \psi \in H \quad (j = 1, \ldots, l_i) \tag{2.6}$$

Lemma 2.1. The conditions (2.4) are necessary in order to solve the inhomogeneous problem.

Proof. Let the minimum of the functional (2.5) be reached on the function $w_0 \in H$; then writing the equation in variations for the functional $F(w)$ we obtain

$$(w_0, \varphi)_H - \nu_i (w_0, \varphi)_{H_\rho} - \iint_D q^2\mathbf{f}\cdot\nabla\varphi\,dx\,d\eta = 0, \qquad \varphi \in H \tag{2.7}$$

Substituting into equation (2.7) the eigenfunction z_{ij} $(j = 1, \ldots, l_i)$ instead of the arbitrary function ϕ, and substituting into equation (2.6) the quantity w_0 instead of the arbitrary function ψ, we obtain conditions (2.4).

Lemma 2.2. The function on which the minimum of the functional $F(w)$ is reached can be represented in the form

$$w_0(x, \eta) = -\sum_{k=1}^{i-1}\sum_{j=1}^{l_k} \frac{z_{kj}(x, \eta)}{\nu_i - \nu_k} \iint_D q^2\mathbf{f}\cdot\nabla z_{kj}\,dx\,d\eta + \sum_{j=1}^{l_i} c_{ij} z_{ij}(x, \eta) + A_i\,\mathbf{f} \tag{2.8}$$

Here $\phi_i = A_i\mathbf{f}$ is the function which produces a minimum of the functional $F(w)$ on the orthogonal addition $H_{\infty - i}$ to the invariant subspace H_i corresponding to the eigenvalues $\nu_1, \ldots, \nu_i$, where c_{ij} $(j = 1, \ldots, l_i)$ are arbitrary numbers.

Proof. Any function $w \in H$ can be represented in the form of a sum $w = w_1 + w_2$, where $w_1 \in H_i$, and $w_2 \in H_{\infty - i}$. But by virtue of orthogonality $F(w) = F(w_1) + F(w_2)$. Since the functions w_1 and w_2 are independent, it is sufficient to find minima

$F(w_1)$ and $F(w_2)$ for finding the minimum $F(w)$. The minimum of a quadratic functional on a finite subspace is found in the elementary way. It is achieved on the function which is obtained from equation (2.8) if $A_i f \equiv 0$ is introduced in this equation. For the existence of a solution to the second problem it is sufficient for the quadratic functional entering into the structure of Fw to be positively defined [3] on the subspace $H_{\infty - i}$.

But this is so because

$$\|w\|_H^2 - \nu_i \|w\|_{H_\rho}^2 \geqslant (\nu_{i+1} - \nu_i)\|w\|_{H_\rho}^2 \qquad (w \in H_{\infty - i})$$

Lemma 2.2. has been proved. Using the same considerations as in [3] it is not difficult to obtain the following evaluations

$$\|A_i f\|_H \leqslant C_1 \|\mathbf{f}\|_L, \quad \|A_i \mathbf{f}\|_{H_\rho} \leqslant C_2 \|\mathbf{f}\|_L, \quad \|\mathbf{f}\|_L = \int_0^1 \int_0^L q^2 |\mathbf{f}|^2 \, dx \, d\eta \tag{2.9}$$

It remains to be shown that the function $w_0(x, \eta)$ on which the minimum of the functional is achieved belongs to H_2, if $f_1 \in B_1$, and $f_2 \in B_1$. An integral representation will be derived from which this will follow.

We construct the following function

$$G(x, \xi, \eta, t) = \frac{2}{L} \sum_{n=1}^{\infty} g\left(\eta, t, \frac{n\pi}{L}\right) \sin \frac{n\pi x}{L} \sin \frac{n\pi \xi}{L} \tag{2.10}$$

$$\left.\begin{array}{l} u = u_k(\eta, \lambda) \\ v = v_k(\eta, \lambda) \end{array}\right\} \quad (h_{k-1} \leqslant \eta \leqslant h_k), \qquad g(\eta, t, \lambda) = \frac{1}{v(0, \lambda)} \begin{cases} u(\eta, \lambda)\, v(t, \lambda) & (\eta \leqslant t) \\ u(t, \lambda)\, v(\eta, \lambda) & (\eta \geqslant t) \end{cases}$$

where the functions $u_k(\eta, \lambda)$ and $v_k(\eta, \lambda)$ are determined by the recurrent formulae

$$u_1(\eta, \lambda) = \frac{\sinh \lambda\eta}{\lambda}, \quad v_n(\eta, \lambda) = \cosh \lambda(\eta - 1), \quad \alpha_k = \frac{q^2(h_k + 0)}{q^2(h_k - 0)}, \quad \alpha_0 = 1, \alpha_n = 0$$

$$u_{k+1}(\eta, \lambda) = u_k(h_k, \lambda) \cosh \lambda(\eta - h_k) + \alpha_k \lambda^{-1} u_k'(h_k, \lambda) \sinh \lambda(\eta - h_k) \qquad (k = 1, \ldots, n-1)$$

$$_k(\eta, \lambda) = v_{k+1}(h_k, \lambda) \cosh \lambda(\eta - h_k) + (\alpha_k \lambda)^{-1} v_{k+1}'(h_k, \lambda) \sinh \lambda(\eta - h_k) \tag{2.11}$$

Lemma 2.3. As $\lambda \to +\infty$ for functions $u_k(\eta, \lambda)$ and $v_k(\eta, \lambda)$, the following asymptotic equations are applicable

$$u_k(\eta, \lambda) = (2^{k-1}\lambda)^{-1}(1 + \alpha_1) \ldots (1 + \alpha_{k-2}) [(1 + \alpha_{k-1}) \sinh \lambda\eta +$$
$$+ (1 - \alpha_{k-1}) \sinh \lambda(2h_{k-1} - \eta)] + O(e^{(1-\varepsilon)\lambda\eta})$$

$$v_k(\eta, \lambda) = \frac{(1 + \alpha_{k+1}) \ldots (1 + \alpha_{n-1})}{2^{n-k} \alpha_k \ldots \alpha_{n-1}} [(1 + \alpha_k) \cosh \lambda(1 - \eta) -$$
$$- (1 - \alpha_k) \cosh \lambda(1 + \eta - 2h_k)] + O(e^{(1-\varepsilon)\lambda\eta}) \tag{2.12}$$

where ϵ is some positive number independent of λ.

The proof is made by induction. For $u_1(\eta, \lambda)$ equation (2.12) is apparently applicable. Let it be applicable for $u_k(\eta, \lambda)$. Utilizing equation (2.11) it is not difficult to show that it is also applicable to $u_{k+1}(\eta, \lambda)$.

Lemma 2.4. If $h_{k-1} \leqslant \eta \leqslant h_k$, and $h_{k-1} \leqslant t \leqslant h_k$, then the function $g(\eta, t, \lambda)$ can be represented in the form

$$g(\eta, t, \lambda) = \frac{\alpha_1 \ldots \alpha_{k-1}}{2\lambda}\left[e^{-\lambda|t-\eta|} + \frac{1-\alpha_{k-1}}{1+\alpha_{k-1}} e^{-\lambda|2h_{k-1}-\eta-t|} - \frac{1-\alpha_k}{1+\alpha_k} e^{-\lambda|2h_k-\eta-t|}\right] + g_1(\eta, t, \lambda) \quad (2.13)$$

In addition to this

$$g(\eta, t, \lambda) = \frac{\alpha_1 \ldots \alpha_k}{2\lambda(1+\alpha_k)} e^{-\lambda|t-\eta|} + g_2(\eta, t, \lambda) \qquad (h_{k-1} \leqslant \eta \leqslant h_k,\ h_k \leqslant t \leqslant h_{k+1})$$

$$g(\eta, t, \lambda) = g_3(\eta, t, \lambda) \qquad (h_{k-1} \leqslant \eta \leqslant h_k,\ h_j \leqslant t \leqslant h_{j+1},\ j \neq k, k+1, k-1)$$

where for the functions $g_i(\eta, t, \lambda)$ and for their derivatives the following asymptotic equations are applicable as $\lambda \to +\infty$

$$\frac{\partial^m g_i}{\partial t^k \partial \eta^{m-k}} = \lambda^{m-1} O(e^{-\varepsilon\lambda}), \qquad \varepsilon > 0 \qquad (m \geqslant k,\ i = 1, 2, 3)$$

Proof of this lemma is easy to obtain if the expression (2.12) is substituted into equation (2.10) for the function $g(\eta, t, \lambda)$.

Lemma 2.5. The function $G(x, \xi, \eta, t)$ is symmetric with respect to the variables x, η and ξ, t, it is harmonic with respect to x and η when $(x, \eta) \in D_k$, $x \neq \xi$, $\eta \neq t$ and it satisfies the following conditions

$$(G)_{x=0} = (G)_{x=L} = (G)_{\eta=0} = [G]_k = 0 \quad (k = 1, \ldots, n-1), \qquad [q^2 G_\eta]_k = 0 \quad (k = 1, \ldots, n) \quad (2.14)$$

If $(\xi, t) \in D_k$, then in the neighborhood of this point the function $G(x, \xi, \eta, t)$ can be represented in the following form

$$G(x, \xi, \eta, t) = \frac{1}{\pi}\alpha_1 \ldots \alpha_{k-1}\left\{\log[(x-\xi)^2 + (\eta-t)^2] + \frac{1-\alpha_{k-1}}{1+\alpha_{k-1}}\log[(x-\xi)^2 + (2h_{k-1}-\eta-t)^2] - \frac{1-\alpha_k}{1+\alpha_k}\log[(x-\xi)^2 + (2h_k-\eta-t)^2]\right\} + G_1(x, \xi, \eta, t), \quad (x, \eta) \in D_k$$

$$G = \frac{\alpha_1 \ldots \alpha_k}{\pi(1+\alpha_k)}\log[(x-\xi)^2 + (\eta-t)^2] + G_1(x, \xi, \eta, t), \qquad (x, \eta) \in D_{k+1} \quad (2.15)$$

$$G = G_3(x, \xi, \eta, t), \qquad (x, \eta) \in D_i \qquad (i \neq k, k-1, k+1)$$

Here $G_i(x, \xi, \eta, t)$ are bounded harmonic functions.

Proof. Let $(\xi, t) \in D_k$. For $\eta \neq t$, as follows from expression (2.13), the series (2.10) can be differentiated term by term and consequently the series will represent a harmonic function. For $\eta = t$, and $x \neq \xi$ the series (2.10) converges. Therefore the function $G(x, \xi, \eta, t)$ will satisfy Laplace's equation at all points of the rectangle D_k with the exception of the point $x = \xi$, $\eta = t$. It is not difficult to verify that this function also satisfies the boundary conditions (2.14). In order to isolate singularities of the function $G(x, \xi, \eta, t)$, it is necessary to substitute expressions (2.13) into the series (2.10). Isolating singularities for sums of corresponding series we obtain equation (2.15). Lemma 2.5 has been proved.

Lemma 2.6. For a function $w_0(x, \eta)$, which produces a minimum of the functional (2.5), the following integral representation is applicable

$$w_0(\xi, t) = -\frac{1}{q^2(t)} \alpha_1 \ldots \alpha_{k-1} \left\{ \iint\limits_D \left[\left(2q \frac{dq}{d\eta} - \nu_i \rho \right) \frac{\partial G}{\partial \eta} w_0 - \right. \right.$$
$$\left. \left. - \nu_i G \frac{\partial w_0}{\partial \eta} - 2q^2(\eta)\, \mathbf{f} \cdot \nabla G \right] dx d\eta \right\}, \qquad (\xi, t) \in D_k \qquad (k = 1, \ldots, n) \tag{2.16}$$

Proof. Using standard reasoning it is not difficult to show that the integral representation (2.16) is applicable to solutions of the boundary value problem (2.3). The function $w_0(x, \eta)$ can be approximated in $W_2^{(1)}$ to any degree of accuracy by a sequence of twice continuously differentiable functions $w_n(x, \eta)$ (for example by the Ritz method). The functions $w_n(x, \eta)$ will be solutions of the sequence of boundary value problems (2.3) where instead of f in the right-hand parts we have $\mathbf{f}_n$, where $\mathbf{f}_n \to \mathbf{f}$ in $W_2^{(1)}$. We write for the function $w_n(x, \eta)$ the integral representation (2.16) and pass to the limit as $n \to \infty$. We obtain that the integral representation (2.16) is applicable for generalized solutions. The lemma has been proved.

Now it is not difficult to complete the proof of theorem 2.2. From the integral representation (2.16) and from the properties of the potentials it follows that the function $w_0(x, \eta)$ has second generalized derivatives in D_k and consequently satisfies Hoelder condition. Corresponding evaluations for the norm $w_0(x, \eta)$ are obtained by the usual method [3] using the evaluations (2.9).

3. Nonlinear theory of small amplitude waves. Let ν_0 be an m-fold eigenvalue of a linear problem and $z_1(x, \eta), \ldots, z_m(x, \eta)$ the eigenfunctions corresponding to this eigenvalue. Taking into account that

$$\rho'(\eta)\, w = \frac{\partial}{\partial \eta} \left[\rho w - \int_1^{\eta} \rho\, w_\eta' d\eta \right]$$

and that the function

$$\int_1^{\eta} \rho w_\eta' d\eta$$

is continuous, and assuming that $\nu = \nu_0 - \mu$, we rewrite equations (1.11) in the form

$$\begin{gathered} Mw - \nu_0\rho'(\eta)\, w = \operatorname{div}\,(q^2 Fw) \\ w\,(x, 0) = w\,(0, \eta) = w\,(L, \eta) = [w]_k = 0 \qquad (k = 1, \ldots, n-1) \\ [q^2 w_\eta' - \nu_0 \rho w - q^2 F_2 w]_k = 0 \qquad (\nu = \nu_0 - \mu,\ k = 1, \ldots, n) \end{gathered} \tag{3.1}$$

$$F_2 w = \Phi_2 w - \mu\rho q^{-2} w + \mu q^{-2} \int_1^\eta \rho w_\eta' d\eta, \qquad Fw = (\Phi_1 w, F_2 w) \tag{3.2}$$

where the nonlinear operators $\Phi_1 w$ and $\Phi_2 w$ are defined by equations (1.12). As follows from theorem 2.2, in order for a solution of the nonlinear boundary value problem to exist it is necessary that the following conditions be satisfied

$$\Omega_i Fw \equiv \iint_D q^2\,(\eta)\; Fw \cdot \nabla z_i dx d\eta = 0 \tag{3.3}$$

We will examine simultaneously the systems of equations (3.1) and (3.3). Applying theorem 2.2 we obtain

$$w = A\left[Fw - \sum_{i=1}^{m} \frac{z_i\,(x, \eta)}{\Omega_i z_i}\, \Omega_i Fw\right] + \sum_{i=1}^{m} c_i z_i\,(x, \eta) \tag{3.4}$$

The boundary value problem (3.1) is equivalent to the solution of the system of equations (3.3) and (3.4). However, equation (3.4) may be solved independently of equations (3.3).

Theorem 3.1. Such numbers $\mu_0 > 0$ and $\alpha_0 > 0$ exist that for $|\mu| < \mu_0$ and $|\alpha_i| < \alpha_0$ in space H_2 a sphere can be found of such a radius ϵ and with center at 0 that in this sphere a solution of equation (3.4) can be obtained by the method of successive approximations and this solution will be an analytical function of the parameters μ, $c_1, \ldots, c_m$.

The proof is omitted because it can be carried out by standard considerations. The analytical behavior, for example, is proved by constructing majorant series [4].

If the solution obtained is now substituted into equation (3.3), we obtain a system of equations with Liapunov-Schmidt branching

$$R_i\,(\mu, c_1, c_2, \ldots, c_m) = 0 \qquad (i = 1, \ldots, m) \tag{3.5}$$

Here R_i are some analytical functions of their own arguments. While complete analysis of the branching equations in the case of a multiple eigenvalue is possible in principle, it leads to extremely cumbersome calculations [5]. We limit ouiselves to the case of a simple eigenvalue. The solution (3.4) has the form

$$w(x, \eta) = \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} w_{ik}(x, \eta) \mu^i \beta^k, \qquad w_{01}(x, \eta) = z(x, \eta) \tag{3.6}$$

Here β is an arbitrary parameter and $z(x, \eta)$ is the eigenfunction. Substituting expression (3.6) into the branching equation (3.3), we obtain

$$R(\mu, \beta) = 0 \tag{3.7}$$

Let

$$\varphi(\zeta) = \Phi\Big[\sum_{k=1}^{\infty} w_{0k}(x, \eta) \zeta^k\Big], \quad \iint_D q^2 \varphi \cdot \nabla z dx d\eta = \sum_{k=2}^{\infty} a_k \zeta^k \tag{3.8}$$

Theorem 3.2. If a_p is the first coefficient different from zero of the series (3.8) and p is even, then the branching equation has for β a single nontrivial solution which can be found in the form of a series in powers of the parameter $\mu^{1/(p-1)}$. If, however, p is odd, then the branching equation has two nontrivial solutions, which for $a_p > 0$ are in the form of series in powers of $\mu^{1/(p-1)}$, and for $a_p < 0$ in the form of series in powers of $(-\mu)^{1/(p-1)}$.

*Proof.*Rememberingexpressions (3.2) and utilizing equations (3.8) we can write branching equations in the form

$$R(\mu, \beta) \equiv a_p \beta^p - b\beta\mu + \sum_{k=p+1}^{\infty} a_k \beta^k + \beta\mu^2 R_1(\mu, \beta) + \beta^2 \mu R_2(\mu, \beta) = 0 \tag{3.9}$$

Here R_1 and R_2 are some analytical functions and

$$b = \int_0^1 \int_0^L \Big[\rho z - \int_0^\eta \rho \frac{dz}{d\eta} d\eta\Big] \frac{dz}{d\eta} dx d\eta = \| z \|_{H_\rho}^2$$

It is known that all solutions of the equation of the form (3.9), which become zero for $\mu = 0$, can be found in the form of series [6] in terms of fractional powers of the parameter μ

$$\beta = \sum_{k=1}^{\infty} \beta_k \mu^{k/r} \tag{3.10}$$

Here r is some integer. Substituting the expansion (3.10) into equation (3.9) we obtain that the terms of lowest power are

$$a_p \beta_1^p \mu^{p/r} - b\beta_1 \mu^{1+1/r}$$

For their mutual annihilation the following conditions must be fulfilled

$$r = p - 1, \qquad a_p \beta_1^p - b\beta_1 = 0$$

Only real solutions of the second equation are of interest. The branching equation has just as many solutions as this equation. Three cases are possible:

(*a*) p is even, there is one nontrivial solution

$$\beta_1 = (b / a_p)^{1/(p-1)}$$

(*b*) p is odd, $a_p > 0$, there are two solutions for $\mu > 0$

$$\beta_1 = \pm (b / a_p)^{1/(p-1)}$$

and there are no solutions for $\mu < 0$;

(*c*) p is odd, $a_p < 0$. This case is reduced to the previous case, if one writes $\mu' = -\mu$ and looks for solutions in the form of a series in fractional powers of parameter μ'. Theorem 3.2 has been proven.

We note that generally speaking $p = 2$ and $a_2 \neq 0$. We will call the case $p = 2$ the general case, and cases $p > 2$ exceptional cases.

Substituting series (3.10) for β into expression (3.6), we obtain solutions of the nonlinear boundary value problem (3.1) in the form of some series in fractional powers of the parameter μ. Remembering equation (1.10) for the family of streamlines and utilizing expression (2.1) for eigenfunctions with $k = 1$, we obtain

$$y(x, \eta) = \eta + A_i\left[1 - \cos\frac{\pi x}{L}\right]u_i\left(\eta, \frac{\pi}{L}\right) + o(A_i)$$

$$A_i = \frac{L}{\pi}\left(\frac{\mu b}{a_p}\right)^{\frac{1}{p-1}} \qquad (p = 2q), \qquad A_i = \pm\frac{L}{\pi}\left(\frac{\mu b}{|a_p|}\right)^{\frac{1}{p-1}} \qquad (p = 2q + 1)$$

It is known that for a rectilinear bed the velocity of wave propagation is an uncertain quantity. In section 1 the quantity $P / H\sqrt{\rho^\circ}$ was referred to as the velocity of propagation. Since $\nu = gH / c^2$, then

$$c_i^2 = \frac{gH}{\nu_i(\pi/L)}\left[1 + \frac{a_p}{b}\left(\frac{A_i\pi}{L}\right)^{p-1}\frac{1}{\nu_i(\pi/L)}\right] + o(A_i^{p-1}) \qquad (3.11)$$

It follows from expression (3.11) that for $\operatorname{sgn}(a_p A_i^{p-1}) > 0$ the velocity of propagation exceeds the critical velocity and grows with increasing amplitude, and for $\operatorname{sgn}(a_p A_i^{p-1}) < 0$ it is smaller than the critical velocity and decreases with increasing amplitude.

The principal result can now be formulated in the following way: an even set (for $\rho'(\eta) \equiv 0$ it is finite) of critical values exists for the wave propagation velocity. If the propagation velocity is close to one of the critical velocities, then, in addition to one-dimensional flow with given density and velocity distribution over the cross section, there always exists, for $a_p \neq 0$ and p eve one family of two-dimensional flows, which for a fixed value of wavelength depends on one nondimensional parameter (amplitude) and which has the same distribution of the density and of the horizontal component of velocity vector over the cross section, which appears as an axis of symmetry, as the one-dimensional flow. For uneven p two families of two-dimensional flows exist propagating with the same velocity. The dependence of this velocity on the amplitude is expressed by equation (3.11).

In the case of problem B everything is also reduced to the analysis of branching equations which have the same form as equation (3.7). But here $p = 3$ in the general case. We present the corresponding equations for the family of streamlines and for the velocities of propagation

$$y\,(x,\,\eta) = \eta + A_i \cos\left(\frac{\pi x}{L}\right) u_i\left(\eta,\,\frac{\pi}{L}\right) + o\,(A_i)$$

$$c_i^2 = \frac{gH}{\nu_i\,(\pi\,/\,L)}\left[1 + a_3 A_i^2\,\frac{1}{b\nu_i\,(\pi\,/\,L)}\right] + o\,(A^2)$$

Here A_i is the amplitude, and a_3 is some known coefficient. The propagation velocity is greater than critical and grows with increase in amplitude for $a_3 > 0$ and is smaller than critical for $a_3 < 0$.

The principal result of problem B can be expressed in the following form: if the velocity of propagation is close to one of the critical velocities, then, in addition to trivial one-dimensional flow with a given distribution of density and average vorticity along streamlines, in the general case two families of two-dimensional flows always exist which for a fixed wavelength depend on one parameter (amplitude) and which have the same distribution of density and average vorticity along streamlines as in the case of one-dimensional flow. In exceptional cases there may not be two, but one family of two-dimensional flows.

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Translated by J.R.W.